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Note on the Solution by Least Squares of the Equations arising in the Reduction of Photographic Star-plates. By Harold Jacoby.

In the determination of the constants of a photographic plate from the measured coordinates of known stars on the plate, we have to carry out the solution of a set of linear equations by the method of least squares or otherwise. The coefficients in these equations possess a peculiarity which will enable us to effect a rigorous least square solution with very little labour. If we indicate by p and r the scale value and orientation constants, and by k and c the two errors of centring, we shall have from the right ascensions of the known stars a series of equations of the form

$$ap + br + k + d' = 0.$$
$$\vdots \qquad \vdots$$

From the declinations we get another series of equations as follows :—

$$bp - ar + c + d = 0.$$
$$\vdots \qquad \vdots$$

If the number of stars be n, and if we indicate the summation of n quantities in the usual way by means of square brackets, the rigorous least square solution of the above $2n$ equations is given by the following formulæ :—

$$A = [aa] - \frac{[a]^2}{n},$$

$$C = [ad'] - \frac{[a]\,[d']}{n},$$

$$D = [bb] - \frac{[b]^2}{n},$$

$$E = [bd'] - \frac{[b]\,[d']}{n},$$

$$C' = [bd] - \frac{[b]\,[d]}{n},$$

$$E' = -[ad] + \frac{[a]\,[d]}{n},$$

$$p = -\frac{C + C'}{A + D}, \qquad \text{weight of } p = A + D,$$

$$r = -\frac{E + E'}{A + D}, \qquad \text{weight of } r = A + D,$$

$$k = -\frac{1}{n}\left\{[a]\,p + [b]\,r + [d']\right\}, \text{ weight of } k = n - \frac{[a]^2 + [b]^2}{[aa] + [bb]},$$

$$c = -\frac{1}{n}\left\{[b]\,p - [a]\,r + [d]\right\}, \text{ weight of } c = n - \frac{[a]^2 + [b]^2}{[aa] + [bb]}.$$

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With the help of a Thomas or Brunsviga calculating machine, these formulæ enable us to perform a least square solution in a very short time indeed. As an example, let us take the following set of equations, which occurred in the discussion of a plate taken by M. Henry at Paris, 1891 December 2.

Equations derived from the Right Ascensions.

$a \quad b \quad d'$	*Res'l.*
$-37p+15r+k+3''\cdot03=0$	$+0''\cdot94$
$-33p-2r+k+2\cdot69=0$	$+0\cdot31$
$-24p+13r+k+1\cdot56=0$	$-0\cdot68$
$-19p+11r+k+1\cdot63=0$	$-0\cdot68$
$-16p+16r+k+2\cdot51=0$	$+0\cdot24$
$-p-3r+k+1\cdot93=0$	$-0\cdot74$
$+2p-27r+k+4\cdot16=0$	$+1\cdot11$
$+2p-1r+k+1\cdot97=0$	$-0\cdot70$
$+24p\quad 0r+k+1\cdot88=0$	$-0\cdot97$
$+32p+33r+k+3\cdot16=0$	$+0\cdot72$
$+34p+17r+k+3\cdot16=0$	$+0\cdot46$

Equations derived from the Declinations.

$b \quad -a \quad d$	*Res'l.*
$+15p+37r+c-15''\cdot15=0$	$-0''\cdot51$
$-2p+33r+c-14\cdot56=0$	$+0\cdot18$
$+13p+24r+c-15\cdot44=0$	$-0\cdot97$
$+11p+19r+c-13\cdot08=0$	$+1\cdot34$
$+16p+16r+c-14\cdot62=0$	$-0\cdot29$
$-3p+r+c-13\cdot41=0$	$+0\cdot87$
$-27p-2r+c-14\cdot77=0$	$-0\cdot32$
$-p-2r+c-14\cdot09=0$	$+0\cdot13$
$0p-24r+c-14\cdot46=0$	$-0\cdot57$
$+33p-32r+c-13\cdot19=0$	$+0\cdot29$
$+17p-34r+c-13\cdot73=0$	$-0\cdot13$

We have here—

$n=11,$

$[a]=-36,$ $[b]=+72,$ $[d']=+27\cdot68,$ $[d]=-156\cdot50$

$[aa]=+6416,$ $[ad']=-\ 45\cdot44,$ $[bb]=+2892,$ $[bd']=+156\cdot36$

$[bd]=-992\cdot22,$ $[ad]=+613\cdot78.$

Consequently—

$A=+6298\cdot18,$ $C=+45\cdot15,$ $D=+2420\cdot73,$ $E=-\ 24\cdot82,$

$C'=+32\cdot14,$ $E'=-101\cdot61,$

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And therefore—

$p = -\ {\cdot}0089$, weight 8719.
$r = +\ {\cdot}0145$, „ 8719.
$k = -\ 2''{\cdot}64$, „ 10·3
$c = +14{\cdot}24$, „ 10·3.

If we substitute these values of the unknowns in the original equations, we get the series of residuals given above. The sum of the squares of the 22 residuals is 10·3179, so that we have—

Mean error of one equation $= \pm 0''{\cdot}76$,
Mean error of p or r $= \pm 0{\cdot}0081$,
Mean error of k or c $= \pm 0{\cdot}24$.

The above method of computation is so simple, and requires so little time, that there seems to be no reason to substitute some other method for the method of least squares in the reduction of photographic plates. It should be noted, however, that in the foregoing computations all the equations received unit weight. But it might be desirable to give different weights to the equations derived from the declinations and right ascensions. In that case some of the simplicity of the solution here given is lost. But we can still employ a tolerably simple system of formulæ. If we compute, in addition to the auxiliaries already computed, the quantity—

$$\mathrm{B} = [ab] - \frac{[a]\,[b]}{n},$$

and if we assume that the right ascension equations are to receive the weight w, and the declination equations the weight w', we can obtain the values of the unknown quantities, such as they result from a rigorous least square solution, by the following system of formulæ :—

$$\mu_1 = w\mathrm{A} + w'\mathrm{D},$$
$$\mu_2 = (w - w')\mathrm{B},$$
$$\mu_3 = w\mathrm{C} + w'\mathrm{C}',$$
$$\mu_4 = w\mathrm{D} + w'\mathrm{A},$$
$$\mu_5 = w\mathrm{E} + w'\mathrm{E}',$$
$$\mathrm{Q}_1 = \mu_4 - \frac{\mu_2^2}{\mu_1}, \qquad \mathrm{Q}_1' = \mu_1 - \frac{\mu_2^2}{\mu_4},$$
$$\mathrm{Q}_2 = \mu_5 - \frac{\mu_2\mu_3}{\mu_1},$$
$$r = -\frac{\mathrm{Q}_2}{\mathrm{Q}_1}, \text{ weight of } r = \mathrm{Q}_1,$$
$$p = -\frac{\mu_3 + \mu_2 r}{\mu_1}, \text{ weight of } p = \mathrm{Q}_1^{1},$$
$$k = -\frac{1}{n}\Big\{[a]p + [b]r + [d']\Big\}, \text{ weight of } k = wn, \text{ nearly.}$$
$$c = -\frac{1}{n}\Big\{[b]p - [a]r + [d]\Big\}, \text{ weight of } c = w'n, \text{ nearly.}$$

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If now these values of the unknowns are substituted in the original equations, and if we indicate by v and v' the residuals from the right ascension and declination equations respectively, then we must use as the sum of the squares of the residuals for the computation of the mean error of one equation of weight unity, the quantity :—

$$w[vv] + w'[v'v'].$$

In some cases it may seem desirable to compute the unknowns separately from the right ascensions and declinations in order to see if they agree within the limits of their probable errors. In doing this we can make use of the auxiliaries given above. The necessary formulæ are as follows :—

Right Ascension Equations.

$$\nu_1 = \mathrm{D} - \frac{\mathrm{B}^2}{\mathrm{A}}, \qquad \nu_2 = \mathrm{E} - \frac{\mathrm{BC}}{\mathrm{A}},$$

$$r = -\frac{\nu_2}{\nu_1}, \text{ weight of } r = \nu_1,$$

$$p = -\frac{\mathrm{B}}{\mathrm{A}}r - \frac{\mathrm{C}}{\mathrm{A}}, \text{ weight of } p = \nu_3 \textit{ (see below)},$$

$$k = -\frac{1}{n}\left\{[a]p + [b]r + [d']\right\}, \text{ weight of } k = n, \text{ nearly.}$$

Declination Equations.

$$\nu_3 = \mathrm{A} - \frac{\mathrm{B}^2}{\mathrm{D}}, \qquad \nu_4 = \mathrm{E}' + \frac{\mathrm{BC}'}{\mathrm{D}},$$

$$r = -\frac{\nu_4}{\nu_3}, \text{ weight of } r = \nu_3,$$

$$p = +\frac{\mathrm{B}}{\mathrm{D}}r - \frac{\mathrm{C}'}{\mathrm{D}}, \text{ weight of } p = \nu_1 \textit{ (see below)},$$

$$c = -\frac{1}{n}\left\{[b]p - [a]r + [d]\right\}, \text{ weight of } c = n, \text{ nearly.}$$

It will be noticed that the computation of the unknowns from the right ascension and declination equations combined, and treated as a single set of equations, is much less laborious than the determination from either set of equations alone. It follows that when a solution from the two sets of equations has once been effected, the combined solution can be worked out with very little additional trouble. We know of no good reason, *a priori*, why the separate solutions should bring out different values of the unknowns. Only by the discussion of a large number of plates, for each of which the two separate solutions as well as the combined solution have been made, can we settle this question. In performing such solutions, the weights and probable errors of the unknowns must always be worked out. For we should not be justified in drawing a conclusion, unless the differences between the values obtained from the separate solutions were appreciably greater than might reasonably be expected after a consideration of their probable errors.

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Photographs of the Spectra of twenty-three characteristic Helium Stars; also Photographs of the Spectra of six Stars of the Third Magnitude, showing the transitions from type to type.

By F. McClean, LL.D., F.R.S.

The photographs shown represent a series of spectra of twenty-three stars, characterised by lines associated with the new element, helium. They have been taken during the last twelve months in the course of photographing the spectra of the northern stars down to the third magnitude. The group thus selected corresponds almost exactly with Class I.*a* of Lockyer's "Classification of Stellar Spectra" (*Phil. Trans.* December 1892). Helium was not then in question, but soon after its discovery Lockyer further attributed the characteristic spectrum of this class to that element (*Proc. Roy. Soc.* May 9, 1895).

The larger series of spectra mentioned will include about 160 stars. Photographs are shown of the spectra of six stars of the third magnitude, illustrating the transitions from type to type. The types are the original ones of Secchi, but Type I. is subdivided into five sections, including that of helium.

A scale of wave-lengths has been constructed by the usual method which will be sufficient to identify the lines. Dunér's Bands are indicated on the spectrum of *α* Herculis.

The instrument used is a photographic telescope made by Sir Howard Grubb. The object glass is of 12 inches aperture and 11 feet 3 inches focal length; a prism of the same aperture, with a refracting angle of 20°, is mounted in a hinged frame in front of the object glass; the cell of the prism also rotates in the plane of the frame; by this means the adjustment of the prism both in position and either on or off the object glass is effected with facility. The telescope was completed in 1895 May.

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Note.—A copy of the photographs will be deposited in the Library. They include the following stars:

Helium Stars.

1. Orion β	Mag. 1	13. Perseus δ	Mag. 3
2. „ γ	„ 2	14. „ ε	„ 3
3. „ δ	„ 2	15. „ ζ	„ 3
4. „ ε	„ 2	16. Lyra β	„ 3
5. „ ζ	„ 2	17. Pegasus γ	„ 3
6. „ ι	„ 3	18. Ursa Minor η	„ 2
7. „ κ	„ 3	19. Virgo α	„ 1
8. Auriga β	„ 2	20. Scorpio β	„ 2
9. Taurus β	„ 2	21. „ δ	„ 2
10. „ ζ	„ 3	22. „ π	„ 3
11. „ η	„ 3	23. „ σ	„ 3
12. Cassiopeia ε	„ 3		